University of Nevada, Reno

# Novel Derivations of the Metric from the Newtonian Potential and of the Relativistic Quantum Equation

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By

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We recommend that the thesis
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Abstract

I derive the general relativistic metric in terms of Newtonian potentials. The result simplifies addition of metrics. Metric operations are derived. I further hypothesize that the symmetric metric could represent an electromagnetic field as well as a gravitational field. I unify the two fields of electromagnetism and gravity into one symmetric metric.

In the second part of this thesis, I derive a novel relativistic quantum equation. The operators are derived, and the uncertainty relations are found. Solutions are analyzed for the flat-space metric. The relativistic harmonic oscillator and 1/r potentials are also presented. The 1/r potential is useful for studying the motion of a particle orbiting a hydrogen atom or a black hole.

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I prayed and thought until two thesis-worthy ideas came to mind. Thanks to God for the sparks of imagination.

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# Chapter 1

# Introduction

Many physics hobbyists wonder whether gravitational waves affect the time calculated by atomic clocks. Gravitational waves are produced by accelerating masses. These masses produce gravitational waves travelling at a finite speed-the speed of light. As they travel, gravitational waves carry energy as shown by Sir Hermann Bondi's "sticky-bead argument." This energy may be absorbed by the atom and disturb its atomic energy levels. The theory behind the atomic clock relies on the fact that electromagnetic laser radiation evolves the configuration of energy levels in a precise way according to a quantum equation of motion known as the Schrödinger equation. Unfortunately, the Schrödinger equation accepts only classical potentials and not relativistic metrics. I will be exploring Newtonian mechanics, Schrödinger mechanics, Einstienian relativity, and relativistic quantum mechanics. This could get confusing with the word classical referring to anything that is not relativistic quantum mechanics, so I shall use "not quantum" to refer to both Newtonian mechanics and Einstein's relativity and "not relativistic" to refer to Newtonian mechanics and Schrödinger mechanics.

This thesis, "A Novel Derivation of the Metric from the Newtonian Potential and of the Relativistic Quantum Equation", will derive a relativistic formulation of particle quantum mechanics with potentials whose information travels at the speed of light. More specifically, the goal is to create a framework to study the effect of gravitational waves on a relativistic, quantum mechanical electron orbiting a classical, stationary proton. The theory can be applied in the context of gravitational waves on atomic clock atoms. A quantum particle orbiting a classical point particle is hard to visualize even for quantum mechanists. Instead let us consider a more intuitive example.

There is a mass or charge moving along a straight line and above it is a particle with either similar or opposite charge. The mass that is moving along the straight line emits force information to the particle above. When the mass is closest to the particle, it emits a stronger force on that mass. As it moves away, the signal gets weaker. These signals are represented by forces, potentials, or metrics. These signal representations are the input into equations of motion in classical (not quantum) mechanics.

The theory developed in the first part of this thesis provides a theory for the representation of Newtonian classical potentials as metrics of Einstein's classical general relativity. The force is the negative derivative of the potential. Newtonian mechanics assumes that information can travel infinitely fast but with Einstein's theory of relativity Einstein adds to Newton's mechanics by stating the particle and force information can only travel at the speed of light.

Einstein's classical general relativity theory provides the most accurate description of large particles traveling at high speeds close to the speed of light. The theory provides solutions for black holes and prior inexplicable deviations from the Newtonian gravitational motion of planets about the sun. The Pound-Rebka gravitational blue-shift experiment proved that light rays are affected by gravity[1]. The Reasenberg-Shapiro Viking Relativity Experiment collected radio signal data from a spacecraft and demonstrated retardation by gravity[2]. Theoretical calculations of the time deviations of clocks on spacecraft due to gravity and relativity accurately reflected experimental measurements of rocket flight times in the Vessot-Levine experiment[3]. Lunar laser ranging tests put stringent limits on the equivalence principle[4]. The Gravity Probe-B experiment are testing theoretical predictions of the geodesic effect and the frame-dragging effect[5]. Properties of black holes are commonly measured such as the spectrum of the accretion discs[6,7,8]. Finally, experiment and theory show deviations from the fairly accurate Newtonian gravitation theory[9,10]. All experiments so far have shown the accuracy of Einstein's theory of general relativity. I must make only slight modifications to general relativity theory avoiding changes of experimentally proven results.

Mathematicians and physicists found Einstein's metric interpretation of gravity very difficult to find tangible solutions for because the field equations are nonlinear, and differential geometry was just beginning its life. Schwarzchild gave the first solution by considering the weak-field limit of a point mass and comparing geodesic solutions (straightest lines possible) of a general metric to the force of gravity[11]. In the first part of this paper, I will reconsider the metric generated by the gravitational field of a point mass (or any other potential) using the metric-potential formulation. I will give a general formula for a metric in terms of any potential as quantum mechanics is usually in terms of potentials, furthermore no calculus or computation is required in getting a metric from a potential (once the general formula is derived). My metric-potential formulation is a formidable mathematical tool to use on otherwise difficult mass distributions, but what is a metric which evidently is related to both Newtonian forces of weak fields and Newtonian potentials?

A metric of a curved surface represents coefficients of the dot product in that surface. The metric description of a surface allows you to compute geometric quantities on a curved surface in an efficient manner. From Einstein's theory, I accept the following: motion is governed by a metric and particles cannot travel faster than a null particle (a particle with 0 mass) in the curved space-time described by this metric.

I deviate from Einstein's theory in that I require multiple metrics for the description of multiple particle motion. Considering only electrically-neutral, massive particles traveling on top of a gravitational field, it is easy to conjecture that only one gravitational metric is needed to describe the motion of particles by the equivalence principle (the statement of equivalence of the inertial and gravitational mass). Both massive particles and null particles travel atop this gravitational field. With much of the interactions on the macroscopic human level being solely electromagnetic, the observer is fooled into thinking gravity bends space and time itself instead of human bodies and electromagnetic light rays.

If I consider instead the metric of an electromagnetic force, this electromagnetic field can bend the paths of electrically charged particles, but not the paths of neutral particles directly through electrical interactions. With this observation, two metrics are needed, a flat space-time metric for a neutral particle, and another metric taking into account the charged particle's geodesic motion (motion due to a metric).

Adding to this theory, I consider the same situation placed on top of a gravitational field which bends all of the above (including the electrical field thus changing the electromagnetic metric) except the flat background metric by my theory. The gravitational and electromagnetic metric will be used to find the particle's geodesic motion described in flat background coordinates. Once I have the geodesic motion of the particle, I could use the metric for measurements of the charged particle's motion. I shall develop the novel theory of potential-metrics to deal with multiple particles in both gravitational and electromagnetic fields.

I extend Einstein's idea that gravitational motion is described by a symmetric metric caused by "curved space" to electromagnetic fields. I develop a novel approach to the construction of metrics given a Newtonian potential measured by a test particle around zero velocity (velocities much less than the speed of light) with respect to the flat background. This theory requires that flat-background reference-frames exist.

In part I of my thesis, I derive a relativistic theory in which all potentials acting on particles are represented by a metric. I include electromagnetic potentials in my metric as gravity is represented by a metric in Einstein's general relativity. My field theory will then be applied to my novel relativistic quantum theory. My metric will be applied to the relativistic quantum equation as the classical potential energy is applied to the Schrödinger equation, the central equation of Newtonian quantum mechanics.

All of quantum mechanics is stated in terms of potentials. Keeping in touch with quantum mechanics, I shall derive quantum relativity in terms of potential-metrics. I have given an explicit unification of only two of the forces which have the most effect on atomic clocks: the gravitational and the electromagnetic.

In part II of my thesis adds to the theory of Einstein's general relativity and uses the metric as input into the quantum relativistic equation that I derive. The theory utilizes the familiar statistical interpretation of quantum mechanics with the wave function which contains all the information about the particle. The second part of the thesis uses knowledge of quantum mechanics as described in the textbook of Quantum Mechanics by David J. Griffiths whereas the first part uses Newton's and Einstein's theories.

I will compare my derivation of my quantum relativistic equation to the standard equation of spin-less quantum relativistic particles called the Klein-Gordon equation[12,13]. The Klein-Gordon equation is the relativistic version of the Schrödinger equation which could be interpreted as the equation used to find the probability amplitude of a spin 0 particle. I will derive an equation similar to the Klein-Gordon equation based on Einstein's proper-time extremization. This derivation allows accurate descriptions of the operators by comparison with the Hamilton-Jacobi equation operators. I further show that rest mass itself is quantized and it has a conjugate relationship with the proper-time of the particle. I derive the operators of 4-velocity and their uncertainty relations with position as well.

Many derivations of the Schrödinger equation exist in the literature[14,15,16,17]. Most derivations start with the postulate that a particle is a wave by the deBroglie hypothesis. The Klein-Gordon equation was derived from the relativistic energy relation $E^2=m^2+p^2$. Given the mass m, the momentum p and the energy E could then be replaced by their classical Schrödinger operator analogues. This derivation uses the operators for classical momentum as the new operators for relativistic momentum.

Instead, I derive a novel relativistic equation using the classical relativistic action. I omit my derivation of the classical Schrödinger equation and operators which the reader can derive in exactly the same way as my relativistic equation from the classical Lagrangian.

Both parts of my thesis are building up toward the description of gravitational waves on the quantum system of an atom in an atomic clock.

# Part I

# Classical Mechanics

# Chapter 2

# Metric-Potential Formulation

A metric is a mathematical description of a curved space. The metric is used to find dot products in that space. The key to my derivation of the metric in terms of potential energy is the accuracy of Newtonian theory at velocities much less than the speed of light. No experiment on any scale, macroscopic or microscopic, has deviated from Newton's inverse-square law of gravity for velocities much less than the speed of light[18,19,20,21]. I postulate that Newton's theory is correct for all particles near zero velocity in a Cartesian Lorentzian observer's frame and use this as a starting point for my derivation of the metric in terms of potentials. (Note throughout this paper $t$ is in units of meters with reciprocal conversion factor $c$, the speed of light). The metric is expressed in the $(1,-1,-1,-1)$ convention[22]. I require that the proper-time produce the same variation as the classical action near zero speed $\left(\frac{dx}{dt}\right)$:

$$\begin{aligned}\int_{\text{small } v} d\tau &= \int_{\text{small } v} \left(\frac{d\tau}{dt}\right) dt \qquad \text{(Einstein)}\\ &= \int \left(kL + \frac{d}{dt}\Phi\left(t, x, .., x', ...\right)\right) dt \qquad \text{(Newton with gauge).}\end{aligned}$$

Here $L = T - V$, $L$ is the Newtonian Lagrangian per test mass, $T$ is the kinetic energy per test mass $\left(\frac{1}{2}\frac{dx}{dt}^2\right)$, $V$ is the potential energy per test mass, $k$ is a constant and $\Phi$ is a twice differentiable function known as a pure differential in variational calculus. It will have no effect on the motion at low speeds. This test mass I mention is the mass used to find the potential at a specified point; it cancels with the mass appearing in the Lagrangian of a gravitational potential by the equivalence principle.

The correctness of Newton's equations for velocities much less than the speed of light leads me to expand Einstein's metric formulation in a Taylor series about velocity 0 and equate it to the Lagrangian formulation. I will expand $\frac{d\tau}{dt}$ to order $v^2$ where $v = \frac{dx^\alpha}{dx^t}$. I first write the function out explicitly (Notice space but not time is included in the Latin letters, e.g. $a$ and $b$; Iuse this convention throughout. Greek letters will range over space-time. $\eta_{\alpha\beta}$ represents the flat metric.):

$$\frac{d\tau}{dt} = \left(g_{tt} + 2\sum_a g_{ta}x'^a + \sum_{ab} g_{ab}x'^a x'^b\right)^{\frac{1}{2}}.$$

I take the zeroth, first and second derivatives, respectively, with respect to Cartesian Lorentzian background velocity ($x' = \frac{dx}{dt}$):

$$\begin{aligned}
\frac{d\tau}{dt}|_{x'=0} &= g_{tt}^{\frac{1}{2}}, \\
\frac{\partial \frac{d\tau}{dt}}{\partial x'^c}|_{x'=0} &= g_{tt}^{-\frac{1}{2}} g_{tc}, \\
\frac{\partial^2 \frac{d\tau}{dt}}{\partial x'^d \partial x'^c}|_{x'=0} &= -g_{tt}^{-\frac{3}{2}} g_{td}g_{tc} + g_{tt}^{-\frac{1}{2}} g_{cd}.
\end{aligned}$$

Now, I set the Einstein formulation and the Newtonian formulation equal and compare results:

$$\begin{aligned}
& g_{tt}^{\frac{1}{2}} + g_{tt}^{-\frac{1}{2}} \sum_a g_{ta}x'^a + \frac{1}{2}\sum_{ab}\left(-g_{tt}^{-\frac{3}{2}} g_{ta}g_{tb} + g_{tt}^{-\frac{1}{2}} g_{ab}\right)x'^a x'^b + o\left(x'^3\right) \qquad (\textit{Einstein} \text{ near } 0) \\
= \; & k\left(\frac{1}{2}x'^2 + A.x' - V\right) + \frac{d\Phi}{dt} \qquad \text{(Newton with gauge)} \\
= \; & \left(-kV + \frac{\partial \Phi}{\partial t}\right) + \sum_a \left(kA_a + \frac{\partial \Phi}{\partial x^a}\right)x'^a + \frac{1}{2}k\sum_a x'^{a2} \qquad \text{(Newton).}
\end{aligned}$$

Here I expressed the action in a general form $\left(\frac{1}{2}x'^2 + A.x' - V\right)$. $A$ is the vector of coefficients of the Newtonian velocities similar to the magnetic potential per test mass and $V$ being the potential energy per test mass. Setting velocity polynomial coefficients equal and solving for the metric, I obtain the relations summarized below.

**Summary 1** ***General Metric-Potential Formulation***

$$
\begin{aligned}
g_{tt} &= \left(-kV + \frac{\partial \Phi}{\partial t}\right)^2, \\
g_{ta} &= \left(-kV + \frac{\partial \Phi}{\partial t}\right)\left(kA_a + \frac{\partial \Phi}{\partial x^a}\right), \\
g_{aa} &= k\left(-kV + \frac{\partial \Phi}{\partial t}\right) + \left(kA_a + \frac{\partial \Phi}{\partial x^a}\right)^2, \\
g_{ab} &= \left(kA_a + \frac{\partial \Phi}{\partial x^a}\right)\left(kA_b + \frac{\partial \Phi}{\partial x^b}\right).
\end{aligned}
$$

I have thus derived the metric $g_{\alpha\beta}$ from the Lagrangian in the form $k\left(\frac{1}{2}x'^2 + A.x' - V\right) + \frac{d\Phi}{dt}$. Notice that $\Phi$ shall have no effect on the motion near velocities much less than the speed of light. $\Phi$ is called the gauging function.

I can obtain a standard form if I assume the metric is flat at infinity and the potential (both $V$ and $A$) is zero at infinity. With $\frac{\partial \Phi}{\partial t} = 1$, and using the gauge $\frac{\partial \Phi}{\partial x^a} = 0$ I get the desired flatness at infinity. The other constant $k = k_g$ is a fundamental constant which must agree with the numerical results in the Pound-Rebka gravitational blueshift experiment [1]. It is possible that in the case of electromagnetism the constant $k = k_e$ is different from the gravitational blueshift constant, $k = k_g$. The constant, $k = k_g$, is a very small number, I will set it to $-1$ for the sake of discussion and analysis of the theory. In the nonphysical case of $k = -1$, I shall call $k$ the unit blueshift constant. The unit blueshift constant shall be used throughout the rest of this paper. In summary,

**Summary 2** ***Flat Infinity Metric Formulation***

*Given the Newtonian Lagrangian of the form* $L = \frac{1}{2}x'^2 + A.x' - V$*, given the nonphysical unit blueshift constant* $k = -1$*, and assuming that the particle is free at infinity; the metric* *is:*

$$
\begin{aligned}
g_{tt} &= (V+1)^2\,, \\
g_{ta} &= -A_a\,(V+1)\,, \\
g_{aa} &= -\,(V+1)+(A_a)^2\,, \\
g_{ab} &= A_a A_b, \\
V_{\vec{r}\to\infty} &= 0.
\end{aligned}
$$

Notice that the potential is potential energy per test mass.

I write out the metric explicitly, remembering $d\tau^2 = dx^\alpha g_{\alpha\beta} dx^\beta$ and $L = \frac{1}{2}x'^2 + A.x' - V$:

$$
g_{\alpha\beta} = \begin{bmatrix}
(V+1)^2 & -A_x\,(V+1) & -A_y\,(V+1) & -A_z\,(V+1) \\
-A_x\,(V+1) & -\,(V+1)+(A_x)^2 & A_x A_y & A_x A_z \\
-A_y\,(V+1) & A_x A_y & -\,(V+1)+(A_y)^2 & A_y A_z \\
-A_z\,(V+1) & A_x A_z & A_y A_z & -\,(V+1)+(A_z)^2
\end{bmatrix}.
$$

# Chapter 3

# Normalized Coordinates

The above formulation gave the metric $g_{\alpha\beta}$ in terms of the Cartesian coordinates of flat space background. In this section, I derive a formula for the transformation of one curvilinear coordinate system to another curvilinear coordinate system; this standard result has been obtained or presented in many texts[23]. This formula will then be used to find the "straight line paths" of particles through space as envisioned by Einstein and described in any modern general relativity textbook[23,24,25,26]. The formulation will prove fruitful, providing a method of obtaining any metric which combines two different types of forces. Specifically, I shall be concerned with the combination of gravity and electromagnetism.

## 3.1 Coordinate Transformation Formula

For any metric $g_{\alpha\beta}$ on coordinates $x^{\alpha}$, my aim is to be able to transform the metric to another set of coordinates, $\xi^{\gamma}$. In other words, my aim is to be able to relabel the space. Let me start with the formula of proper-time:

$$\tau = \int \sqrt{x^{\alpha\prime} g_{\alpha\beta} x^{\beta\prime}} d\tau.$$

My coordinate transformation must preserve the proper-time of the particle since it is the particle's own time. The path taken through space shall also remain the same regardless of the labelling of the coordinates. Both conditions are satisfied if I perform a calculus transformation of variables. Letting $h_{\gamma\delta}$ be the new metric and comparing the two formulas:

$$\tau = \int \sqrt{\left(\xi^{\gamma\prime}\frac{\partial x^{\alpha}}{\partial \xi^{\gamma}}\right) g_{\alpha\beta}\left(\frac{\partial x^{\beta}}{\partial \xi^{\delta}}\xi^{\delta\prime}\right)} d\tau = \int \sqrt{\xi^{\gamma\prime} h_{\gamma\delta}\xi^{\delta\prime}} d\tau.$$

I see that,

$$h_{\gamma\delta} = \frac{\partial x^{\alpha}}{\partial \xi^{\gamma}} g_{\alpha\beta}\frac{\partial x^{\beta}}{\partial \xi^{\delta}},$$

where $\xi^{\delta\prime} = \frac{d\xi^{\delta}}{d\tau}$.

I will now demonstrate a few examples. The first example will get us a general result - the metric potential formulation in spherical coordinates. The second example will get us the metric potential formulation in cylindrical coordinates. In the third example, I will give the derivation of a relativistically corrected current carrying charged wire.

Example: Transform the potential metric formulation from the Cartesian coordinates $x^{\alpha} = (t, x, y, z)$ into the spherical coordinates $\xi^{\gamma} = (t, r, \theta, \phi)$ for a spherically symmetric metric independent of the vector potential. Here $\theta$ is the azimuthal angle, $\phi$ is the polar angle, and $r$ is the radius.

First, I need to find $\frac{\partial x^{\alpha}}{\partial \xi^{\gamma}}$ as a function of the new coordinates, $\xi^{\gamma}$. The coordinate transformation from rectangular to spherical can be found on the Mathematica website[27], for example. I shall multiply both sides of the metric in rectangular coordinates by $\frac{\partial x^{\alpha}}{\partial \xi^{\gamma}}$ and $\left(\frac{\partial x^{\alpha}}{\partial \xi^{\gamma}}\right)^{T}$.

$$\begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ 0 & \sin\phi\sin\theta & r\cos\phi\sin\theta & r\cos\theta\sin\phi \\ 0 & \cos\theta & 0 & -r\sin\theta \end{bmatrix} \begin{bmatrix} dt \\ dr \\ d\phi \\ d\theta \end{bmatrix}.$$

$$h_{\gamma\delta} = \begin{bmatrix} (V+1)^2 & 0 & 0 & 0 \\ 0 & -(V+1) & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta\,(V+1) & 0 \\ 0 & 0 & 0 & -r^2\,(V+1) \end{bmatrix}.$$

Here $V$ is now described in spherical coordinates, $V(r, \phi, \theta)$. From here, I arrive at the following:

**Summary 3** ***Spherical Metric***

*The metric of a spherically symmetric time-independent potential $V(r,\phi,\theta)$ described by spherical coordinates $\xi^\gamma = (t,r,\phi,\theta)$ is,*

$$
\begin{aligned}
g_{tt} &= (V+1)^2 , \\
g_{rr} &= -(V+1) , \\
g_{\phi\phi} &= -r^2 \sin^2\theta \, (V+1) , \\
g_{\theta\theta} &= -r^2 (V+1) .
\end{aligned}
$$

Example: Derive the coordinate transformation from rectangular $x^\alpha = (t,x,y,z)$ to cylindrical coordinates $\xi^\gamma = (t,r,\theta,z)$.

I first write the usual coordinate transformation:

$$
\begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -r\sin\theta & 0 \\ 0 & \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dr \\ d\theta \\ dz \end{bmatrix} .
$$

Now, using the metric in rectangular coordinates with $A = 0$ and multiplying on both sides by the above transformation, I arrive at the following general result:

**Summary 4** ***Cylindrical Metric***

*The metric of a cylindrical time-independent potential, $V(r,\theta,z)$ described by cylindrical coordinates $\xi^\gamma = (t,r,\theta,z)$ is,*

$$
\begin{aligned}
g_{tt} &= (V+1)^2 , \\
g_{rr} &= -(V+1) , \\
g_{\theta\theta} &= -r^2 (V+1) , \\
g_{zz} &= -(V+1) .
\end{aligned}
$$

Example: Use the cylindrical coordinates $x^\alpha = (t,r,\theta,z)$ above and the Lorentz boost transformation formula to derive the magnetic field around a charged wire travelling at speed $v$. The wire is along the $z$ axis and travelling in the $+z$ direction.

The Lorentz transformation in the $+z$ direction for cylindrical coordinates is:

$$\frac{\partial x^\alpha}{\partial \xi^\beta} = \begin{bmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{bmatrix}.$$

Here $\gamma = \frac{1}{\sqrt{1-v^2}}$, and $v$ is the velocity of the observer with respect to the flat background in which the potential was initially described. Then I arrive at:

$$h_{\gamma\delta} = \begin{bmatrix} \gamma^2 (V+1)\left(-v^2+V+1\right) & 0 & 0 & Vv\gamma^2 (V+1) \\ 0 & -(V+1) & 0 & 0 \\ 0 & 0 & -r^2 (V+1) & 0 \\ Vv\gamma^2 (V+1) & 0 & 0 & \gamma^2 (V+1)\left(Vv^2-1+v^2\right) \end{bmatrix}.$$

Consider the metric of a uniformly electrically charged wire. The electric field of the wire is $E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}$, where $\lambda$ is the charge density of wire, $r$ is the distance from the wire and $\varepsilon_0$ is the permittivity of free space[28]. The potential (potential energy per test mass) is therefore $V = -\frac{1}{4\pi\varepsilon_0} 2\lambda \left(\frac{q}{m}\right) \ln r$. Here $\frac{q}{m}$ is the particle's ratio of rest charge $q$ per rest mass $m$, the equivalence ratio of the electromagnetic theory.

Then,

$$\begin{aligned} g_{tt} &= \gamma^2 \left(-\frac{1}{4\pi\varepsilon_0} 2\lambda \left(\frac{q}{m}\right) \ln r + 1\right)\left(-v^2 - \frac{1}{4\pi\varepsilon_0} 2\lambda \left(\frac{q}{m}\right) \ln r + 1\right), \\ g_{tz} &= -\frac{1}{4\pi\varepsilon_0} 2\lambda \left(\frac{q}{m}\right) \ln r * v\gamma^2 \left(-\frac{1}{4\pi\varepsilon_0} 2\lambda \left(\frac{q}{m}\right) \ln r + 1\right). \end{aligned}$$

Now, this metric is also described by background inertial frame coordinates since it is at constant velocity from the first background inertial frame coordinates. I set equal the general metric-potential $g_{tz}$ and the $g_{tz}$ above to find the coefficient $A_z$. I use the rectangular metric $g_{tt}$ and $g_{tz}$ since it is the same as the cylindrical metric in the $t$ and $z$ directions.

Inverting the function $g_{tt}(V)$ I get:

$$V = \sqrt{g_{tt}} - 1.$$

Now comparing with the rectangular metric, $-A_z\,(V+1) = g_{tz}$ therefore:

$$-A_z\left(\sqrt{g_{tt}}\right) = g_{tz}.$$

Solving for the $A_z$ of the charged travelling wire:

$$\begin{aligned} A_z &= -\frac{g_{tz}}{\sqrt{g_{tt}}} = -\frac{-\frac{1}{4\pi\varepsilon_0}2\lambda\left(\frac{q}{m}\right)v\gamma^2\left(-\frac{1}{4\pi\varepsilon_0}2\lambda\left(\frac{q}{m}\right)\ln r + 1\right)\ln r}{\sqrt{\gamma^2\left(-\frac{1}{4\pi\varepsilon_0}2\lambda\left(\frac{q}{m}\right)\ln r + 1\right)\left(-v^2 - \frac{1}{4\pi\varepsilon_0}2\lambda\left(\frac{q}{m}\right)\ln r + 1\right)}} \\ &= -\frac{\lambda}{2\pi\varepsilon_0}\frac{q}{m}(\ln r)\left(v\gamma\sqrt{\frac{\frac{\lambda}{2\pi\varepsilon_0}\frac{q}{m}\ln r - 1}{\left(v^2 + \frac{\lambda}{2\pi\varepsilon_0}\frac{q}{m}\ln r - 1\right)}}\right). \end{aligned}$$

(Note that the magnetic field is $\mathbf{B} = \boldsymbol{\nabla}\times\left(\frac{m}{q}\mathbf{A}\right)$ and not $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Remember that I have defined $L = \frac{1}{2}\frac{d\mathbf{x}}{dt}^2 + \mathbf{A}.\frac{d\mathbf{x}}{dt} - V$, whereas the usual definition is $L = \frac{1}{2}m\frac{d\mathbf{x}}{dt}^2 - qV + q\mathbf{A}.\frac{d\mathbf{x}}{dt}$.) Notice that for $v = 0$, I get $A_z = 0$ as expected. For $\lambda = 0$, I get $A_z = 0$ as expected. This is a relativistic correction to the current through a charged current-carrying wire. The magnetization is a result of the change of reference frame.

If I add a stationary wire of equal but opposite charge as the first charged wire, I get the metric for a neutral wire with a current travelling through it.

## 3.2 Normalized Coordinates

The coordinate transformation derived from above can be used to transform between curvilinear coordinates or reference coordinates labeling the space. In this section, I shall find a very special set of coordinates associated with a given metric $g_{\alpha\beta}$. This set of coordinates shall simplify the calculations of the paths of particles. In fact, all null particle paths in this metric are straight lines. The normalized coordinates are given by the following formula:

**Summary 5** ***The Normalized Coordinates Formula***

$$\frac{\partial x^\alpha}{\partial \xi^\gamma} g_{\alpha\beta} \frac{\partial x^\beta}{\partial \xi^\delta} = \eta_{\gamma\delta}.$$

The $\eta_{\gamma\delta}$ is the flat metric associated with the $x^\beta$ Cartesian, spherical, or other coordinates systems but replaced one for one by the $\xi^\gamma$. In some sense, I have "flattened" the space. I shall call these coordinates, $\xi^\gamma$, the **normalized coordinates** of the metric $g_{\alpha\beta}$.

Example: Find the normalized coordinates of the spherically symmetric time-independent metric using spherical coordinates $(t, r, \phi, \theta)$.

Setting $\frac{\partial x^\alpha}{\partial \xi^\gamma} g_{\alpha\beta} \frac{\partial x^\beta}{\partial \xi^\delta} = \eta_{\gamma\delta}$ I have:

$$\begin{aligned} (V+1)^2 \left(\frac{\partial t}{\partial \xi^0}\right)^2 &= 1, \\ -(V+1)\left(\frac{\partial r}{\partial \xi^1}\right)^2 &= -1, \\ -r^2 \sin^2\theta\,(V+1)\left(\frac{\partial \phi}{\partial \xi^2}\right)^2 &= -\left(\xi^1\right)^2 \sin^2\left(\xi^3\right), \\ -r^2 (V+1)\left(\frac{\partial \theta}{\partial \xi^3}\right)^2 &= -\left(\xi^1\right)^2. \end{aligned}$$

Solving this system of four differential equations, I arrive at:

**Summary 6** ***Spherical Normalized Coordinates***

*The normalized coordinates of the spherically symmetric time-independent metric are:*

$$\begin{aligned} \xi^0 &= (V+1)\,t, \\ \xi^1 &= \int^r \sqrt{(V+1)}dr, \\ \xi^2 &= \frac{r(\sin\theta)\,\phi\sqrt{(V+1)}}{\left(\int^r \sqrt{(V+1)}dr\right)\sin\left(\frac{r\theta\sqrt{(V+1)}}{\int^r\sqrt{(V+1)}dr}\right)}, \\ \xi^3 &= \frac{r\phi\sqrt{(V+1)}}{\int^r \sqrt{(V+1)}dr}. \end{aligned}$$

Example: Find the normalized coordinates of the metric of a point mass outside its black hole radius. Notice that the normalized coordinates do not span the entire space. Metric singularities, especially black hole surfaces, make it necessary to calculate different metrics for different parts of the space.

The potential of the point mass due to gravity is $V = -\frac{GM}{r}$. Using the integral:

$$\int_{GM}^{r} \sqrt{\left(1 - \frac{GM}{r}\right)} dr = GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right),$$

I can find the normalized coordinates of a point mass.

**Summary 7** ***Normalized Coordinates of a Point Mass,*** $M$

$$\begin{aligned}
\xi^0 &= \left(1 - \frac{GM}{r}\right) t, \\
\xi^1 &= GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right), \\
\xi^2 &= \frac{r\,(\sin\theta)\,\phi \sqrt{(1 - \frac{GM}{r})}}{GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right)}, \\
&\quad * \sin \left( \frac{r\theta \sqrt{(1 - \frac{GM}{r})}}{GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right)} \right) \\
\xi^3 &= \frac{r\theta \sqrt{(1 - \frac{GM}{r})}}{GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right)}.
\end{aligned}$$

Notice that the black hole surface has shrunk to a point, the origin. As long as a particle produces a point singularity in the metric, this transformation makes sense mathematically.

I will find some particle path solutions to the point mass metric using the normalized coordinates. The path of a particle falling straight towards the point mass is given by:

$$\begin{aligned}
\tau &= \int_0^{\tau} \sqrt{\left(\xi'^0\right)^2 - \left(\xi'^1\right)^2} d\lambda. \\
\xi^0 &= a\xi^1 + b \text{ for } a, b \text{ constants.}
\end{aligned}$$

Finding the $\xi^\alpha$ from above:

$$\begin{aligned}
\xi^0 &= \left(1 - \frac{GM}{r}\right) t, \\
\xi^1 &= GM \left( \sqrt{\frac{r}{GM}(\frac{r}{GM} - 1)} - \frac{1}{2} \ln \left( 2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM} - 1\right)} \right) \right).
\end{aligned}$$

The general solution along this direction is:

$$t = \frac{aGM\left(\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)} - \frac{1}{2}\ln\left(2\frac{r}{GM} - 1 + 2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right) + b}{\left(1 - \frac{GM}{r}\right)}.$$

# Chapter 4

# Metric Operations

Now that I have a metric in terms of a general potential, a change of variables formula and a normalized coordinate formula. Let me give the operations that can be performed on metrics.

**Summary 8** ***Metric Operations***

*(1) A change of coordinates is permitted by the usual change of coordinates,*

$$h_{\gamma\delta} = \frac{\partial x^\alpha}{\partial \xi^\gamma} g_{\alpha\beta} \frac{\partial x^\beta}{\partial \xi^\delta}.$$

*The metric-potential form will be conserved for Lorentz invariant transformations from background reference frames.*

*(2) Metric binary addition between force fields of the same type (gravity, electromagnetic) is defined so that the net potential of the two metrics is the sum of the two individual Newtonian potentials,*

$$g_{\alpha\beta}(V_1, A_1) \oplus h_{\alpha\beta}(V_2, A_2) = (g \oplus h)_{\alpha\beta} (V_1 + V_2, A_1 + A_2).$$

*(3) Metric binary addition between force fields of different types is defined by the following:*

*Let the force(s) associated with $h_{\alpha\beta}$ metric be the force whose null particle paths are bent by the force(s) associated with $g_{\alpha\beta}$ (e.g. photons are bent by gravitons). Let $\xi^\delta$ be the normalized coordinates of metric $g_{\alpha\beta}$ and let $x^\alpha$ be the natural coordinates used to describe $g_{\alpha\beta}$ above a flat reference,*

$$(g \oplus h)_{\alpha\beta} = \frac{\partial \xi^\delta}{\partial x^\alpha} h_{\delta\gamma} \left(\xi^\delta \left(x^\varepsilon\right)\right) \frac{\partial \xi^\gamma}{\partial x^\beta}.$$

Rule (1) is common in the literature. Rule (2) is from the addition of Newtonian potentials. Rule (3) is from the fact that paths of null particles are bent just as particles are bent by the underlying metric.

I will now demonstrate some examples to find the normalized coordinates of a massive particle.

Example: Use (3) to find the metric of a charged massive particle.

I have already found the metric and normalized coordinates of a spherically symmetric potential with a potential proportional to $\frac{1}{r}$ (Note: The correctness of the electric field metric potential formulation lies in the fact that all particles of the same $\frac{q}{m}$ ratio fall with the same acceleration.):

$$
\begin{aligned}
V &= \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}, \\
g_{00} &= \left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{\xi^1}+1\right)^2, \\
g_{11} &= -\left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{\xi^1}+1\right), \\
g_{22} &= \frac{1}{2}\left(\xi^1\right)^2\left(\cos 2\xi^3-1\right)\left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{\xi^1}+1\right), \\
g_{33} &= -\left(\xi^1\right)^2\left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{\xi^1}+1\right).
\end{aligned}
$$

Now placing this metric atop the gravitational metric of a point particle, I obtain the metric of a charged point mass using metric operation (3) above:

$$
g_{\alpha\beta}\oplus h_{\alpha\beta}=\frac{\partial\xi^\delta}{\partial x^\alpha}h_{\delta\gamma}\left(\xi^\delta\left(x^\alpha\right)\right)\frac{\partial\xi^\gamma}{\partial x^\beta}.
$$

$$
\begin{bmatrix}\frac{d\xi^0}{dt}\\ \frac{d\xi^1}{dr}\\ \frac{d\xi^2}{d\theta}\\ \frac{d\xi^3}{d\phi}\end{bmatrix}
=
\begin{bmatrix}
\left(1-\frac{GM}{r}\right)\\
\sqrt{\left(1-\frac{GM}{r}\right)}\\
\dfrac{r\sqrt{\frac{1}{2}(1-\cos 2\phi)\left(1-\frac{GM}{r}\right)}}{\left(GM\left(\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)\right)\sin\left(\xi^3\right)}\\
\dfrac{r\sqrt{\left(1-\frac{GM}{r}\right)}}{GM\left(\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}
\end{bmatrix}.
$$

$$h_{00}\left(\xi^{\delta}\left(x^{\alpha}\right)\right)$$

$$= \left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}+1\right)^2$$

$$h_{11}\left(\xi^{\delta}\left(x^{\alpha}\right)\right)$$

$$= -\left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}+1\right)$$

$$h_{22}\left(\xi^{\delta}\left(x^{\alpha}\right)\right)$$

$$= \frac{1}{2}\left(GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)\right)^2 *$$

$$* \left(\cos\left(2\left(\frac{r\phi\sqrt{\left(1-\frac{GM}{r}\right)}}{GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}\right)\right)-1\right) *$$

$$* \left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}+1\right).$$

$$h_{33}\left(\xi^{\delta}\left(x^{\alpha\beta}\right)\right)$$

$$= -\left(GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)\right)^2 *$$

$$* \left(\frac{Q}{4\pi\varepsilon_0}\left(\frac{q}{m}\right)\frac{1}{GM\left(\sqrt{\frac{r}{GM}(\frac{r}{GM}-1)}-\frac{1}{2}\ln\left(2\frac{r}{GM}-1+2\sqrt{\frac{r}{GM}\left(\frac{r}{GM}-1\right)}\right)\right)}+1\right)$$

Finally, multiplying the three matrices $\frac{\partial\xi^{\delta}}{\partial x^{\alpha}}h_{\delta\gamma}\left(\xi^{\delta}\left(x^{\alpha}\right)\right)\frac{\partial\xi^{\gamma}}{\partial x^{\beta}}$, I have the metric of a point charge and mass, which is a nice model for a proton. After multiplying by the coordinate transformation expressed in the $x^{\alpha}$ on both sides, the final metric formula becomes very complex and will not aid in intuition, so I omit it.

# Part II

# Quantum Mechanics

# Chapter 5

# Relativistic Equation

I shall derive my relativistic quantum equation from the proper-time action of Einstein. In this section, I again use the conventions, $c = 1$ and $g_{\alpha\beta} = (+, -, -, -)$.

The formula for proper-time is:

$$\tau = \int_a^b \sqrt{g_{\alpha\beta} x^{\alpha\prime} x^{\beta\prime}} d\lambda.$$

Here $\tau$ is the proper-time, $g_{\alpha\beta}$ is the metric, and the $x^{\alpha\prime} = \frac{dx^\alpha}{d\tau}$ are the derivatives of the space-time coordinates $(x, y, z, t)$ with respect to proper-time. I need to find the Hamilton form of this action before I can cast this into the Hamilton-Jacobi form, but the square root in this equation is not very easy to manipulate into the Hamilton form. I shall cast this action into an alternative form presented in [23]. The action to be maximized is:

$$L = \int \frac{1}{2} g_{\beta\gamma} x^{\beta\prime} x^{\gamma\prime} d\tau.$$

Here $g_{\alpha\beta}$ is the metric assumed to be proper-time independent, the $x^{\alpha\prime} = \frac{dx^\alpha}{d\tau}$ are the derivatives of the space-time coordinates with respect to proper-time, and $L$ is no longer the proper-time, $\tau$. After finding the Hamilton-Jacobi form of this action, I then multiply by the probability $|\Psi|^2$ and integrate over all points of background Cartesian Lorentzian flat space-time including time $t$, i.e. $(x, y, z, t)$ to force adherence to these equations of motion. The equation is then:

$$E\Psi = -\frac{1}{2}K^2(-\eta)^{-\frac{1}{2}}\frac{\partial}{\partial x^\delta}\left(\frac{\partial x^\gamma}{\partial x_\neg^\alpha}\frac{\partial x^\delta}{\partial x_\neg^\beta}g_\neg^{\alpha\beta}\sqrt{-\eta}\frac{\partial\Psi}{\partial x^\gamma}\right).$$

Here $\Psi(t, x, y, z)$ is the stationary (in proper-time) wave function corresponding to eigenvalue $E$, $K$ is related to a fundamental constant $\hbar$, $\sqrt{-\eta} = \frac{\partial(x^\alpha)}{\partial\left(x_\neg^\beta\right)}$ is the determinant transformation from the flat background metric, $g_\neg^{\alpha\beta}$ is the previous flat background metric inverse, labeled with Cartesian Lorentzian background coordinates, $x_\neg^\alpha$ are the coordinates of the flat Cartesian background, $x^\gamma$ are the coordinates of the background metric you wish to use and $E = -\frac{1}{2}\mu^2$ where $\mu$ is the quantized rest mass ratio, i.e. the mass of the particle is given by $\mu m_r$ with $m_r$ the rest mass ($\mu$ will be 0 for a massless particle such as a photon or graviton and nonzero for a massive particle). Notice that $\frac{\partial x^\gamma}{\partial x_\neg^\alpha}\frac{\partial x^\delta}{\partial x_\neg^\beta}g_\neg^{\alpha\beta} = g^{\gamma\delta}$ where $g^{\gamma\delta}$ is the metric of the $x^\gamma$.

By the relation $S = K\ln\Psi$, I am able to derive the proper-time dependent quantum equation:

$$-K\frac{\partial}{\partial\tau}\Psi = H\Psi = -\frac{1}{2}K^2(-\eta)^{-\frac{1}{2}}\frac{\partial}{\partial x^\delta}\left(\frac{\partial x^\delta}{\partial x_\neg^\beta}g_\neg^{\alpha\beta}\frac{\partial x^\gamma}{\partial x_\neg^\alpha}\sqrt{-\eta}\frac{\partial\Psi}{\partial x^\gamma}\right).$$

Here $\Psi\left(x^\delta, \tau\right)$ is now the proper-time dependent wave function, $H$ is the Hamiltonian operator represented on the right, $\tau$ is the proper-time of the particle. This derivation of the proper-time dependent equation is explained further in the next section in which I derive the operators. This form of the proper-time dependent equation shows the Lorentz invariance of the equation. A Lorentz boost is absorbed by the change of coordinates formula sandwiched in the middle.

If I further use the fact that $\frac{\partial x^\delta}{\partial x_\neg^\beta}g_\neg^{\alpha\beta}\frac{\partial x^\gamma}{\partial x_\neg^\alpha}$ is a transformation of a metric into the metric $g^{\delta\gamma}$, I obtain the following:

**Summary 9** ***General Quantum Equation***

$-K\frac{\partial}{\partial\tau}\Psi = -\frac{1}{2}K^2(-\eta)^{-\frac{1}{2}}\frac{\partial}{\partial x^\delta}\left(g^{\delta\gamma}\sqrt{-\eta}\frac{\partial\Psi}{\partial x^\gamma}\right)$,

$x_\neg^\alpha(x^\beta)$ *are the rectangular coordinates of flat space,*

$x^\delta$ *are the coordinates with Jacobian from flat Cartesian background* $\sqrt{-\eta} = \frac{\partial\left(x^\delta\right)}{\partial\left(x_\neg^\alpha\right)}$

$g^{\alpha\beta}$ *is the inverse of the metric with Jacobian* $\sqrt{-\eta}$ *from the background rectangular metric.*

For flat Cartesian background coordinates, I have:

$$-K\frac{\partial}{\partial\tau}\Psi = -\frac{1}{2}K^2\frac{\partial}{\partial x_\neg^\beta}\left(g_\neg^{\alpha\beta}\frac{\partial\Psi}{\partial x_\neg^\alpha}\right).$$

# Chapter 6

# Operators

The above equation provides $\Psi\left(x^{\alpha},\tau\right)$ which I normalize across space-time according to the requirement:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi|^{2}\,dt_{\neg}dx_{\neg}dy_{\neg}dz_{\neg}=1.$$

Here again the symbol $^{\neg}$ represents the flat Cartesian Lorentzian coordinate reference frame.

By the relation, $S=K\ln\Psi$, I will find relations for velocities(in proper-time). The classical Hamilton-Jacobi theory tells me how to find momenta from $S$ ($S$ is the solution of the differential equations):

$$\begin{aligned}\frac{dS}{d\tau} &= -H,\\ \frac{\partial S}{\partial x_{\neg}} &= p_{x_{\neg}},\\ x_{\neg}^{\beta\prime} &= g^{\beta\alpha}p_{\alpha},\\ p_{\alpha} &= x'_{\neg\alpha}.\end{aligned}$$

In the statistical interpretation, derivatives of $S$ which where exact momenta quantities in the Hamilton-Jacobi theory now become conditional expectation values:

$$E\left(p_{\neg\alpha}|x_{\neg}^{\gamma}\right)=\frac{\partial S}{\partial x_{\neg}^{\alpha}}=K\Psi^{-1}\frac{\partial\Psi}{\partial x_{\neg}^{\alpha}}.$$

Here $E\left(p_{\neg\alpha}|x_{\neg}^{\gamma}\right)$ is the conditional expectation value of $p_{\neg\alpha}=g_{\alpha\beta}x^{\beta\prime}$ given the point $x_{\neg}^{\gamma}$. Multiplying this by the probability density $\left|\Psi\left(x_{\neg}^{\gamma}\right)\right|^{2}$, I now obtain the expectation value:

$$\begin{aligned} E\left(p_{\neg\alpha}\right) &= \int K\Psi^{-1}\frac{\partial\Psi}{\partial x_{\neg}^{\alpha}}\left|\Psi\right|^{2}dt_{\neg}dx_{\neg}dy_{\neg}dz_{\neg}, \\ &= \int \Psi^{*}K\frac{\partial\Psi}{\partial x_{\neg}^{\alpha}}dt_{\neg}dx_{\neg}dy_{\neg}dz_{\neg}. \end{aligned}$$

Clearly, $\widehat{p_{\neg\alpha}} = K\frac{\partial}{\partial x_{\neg}^{\alpha}}$, and $K$ is imaginary for Hermicity of the observable $\widehat{p_{\neg\alpha}}$. I set $K = -i\frac{\hbar}{m}$, with $m$ the rest mass of the particle. To find the operator for the proper-time velocity, I take half the anticommutator of the two Hermitian operators $g_{\neg}^{\alpha\beta}$ and $\widehat{p_{\neg\beta}}$:

$$\begin{aligned} \widehat{p_{\neg}^{\alpha}} &= \frac{1}{2}\left\{g_{\neg}^{\alpha\beta},\widehat{p_{\neg\beta}}\right\} \\ &= \frac{1}{2}\left(g_{\neg}^{\alpha\beta}\widehat{p_{\neg\beta}}+\widehat{p_{\neg\beta}}g_{\neg}^{\alpha\beta}\right). \end{aligned}$$

Here $g_{\neg\beta\alpha}$ is the Cartesian metric and $\widehat{p_{\neg}^{\alpha}}$ is the proper-time velocity $\left(\frac{dx_{\neg}^{\alpha}}{d\tau}\right)$ operator. With the value $K = -i\frac{\hbar}{m}$, I get the general background coordinate equation:

$$i\frac{\hbar}{m}\frac{\partial}{\partial\tau}\Psi = \frac{1}{2}\left(\frac{\hbar}{m}\right)^{2}(-\eta)^{-\frac{1}{2}}\frac{\partial}{\partial x^{\delta}}\left(g^{\delta\gamma}\sqrt{-\eta}\frac{\partial\Psi}{\partial x^{\gamma}}\right).$$

Summarizing the flat Cartesian background result,

**Summary 10** ***Cartesian Quantum Equation with Operators***

*In Cartesian Lorentzian background coordinates, the operator form of the relativistic equation is:*

$$i\frac{\hbar}{m}\frac{\partial}{\partial\tau}\Psi = \widehat{p_{\tau}}\Psi = -\frac{1}{2}\widehat{p_{\neg\alpha}}g_{\neg}^{\alpha\beta}\widehat{p_{\neg\beta}}\Psi.$$

*The associated operators are:*

$\widehat{p_{\tau}} = i\frac{\hbar}{m}\frac{\partial}{\partial\tau}$ *is interpreted as* $-\frac{\mu^{2}}{2}$ *or* $-\frac{1}{2}x_{\neg}^{\alpha\prime}g_{\neg\alpha\beta}x_{\neg}^{\beta\prime}$.

$\widehat{p_{\neg\alpha}} = -i\frac{\hbar}{m}\frac{\partial}{\partial x_{\neg}^{\alpha}}$ *is interpreted as* $g_{\neg\alpha\beta}x_{\neg}^{\beta\prime}$.

$\widehat{x_{\neg}^{\alpha}} = x_{\neg}^{\alpha}$ *is interpreted as the background coordinate* $x_{\neg}^{\alpha}$.

$\tau$ *is the evolution parameter called the proper-time.*

# Chapter 7

# Uncertainty Relations

I shall derive the uncertainty relations between the momentum operators and space operators. The generalized uncertainty principle derived from the Cauchy-Schwartz inequality is:

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \left\langle \left[ \widehat{A}, \widehat{B} \right] \right\rangle \right)^2 .$$

Remembering that $\widehat{p_{\neg\alpha}}$ is interpreted as $g_{\neg\alpha\beta} x_{\neg}^{\beta\prime}$, and with $\left[ \widehat{x_{\neg}^{\alpha}}, \widehat{p_{\neg\alpha}} \right] = i\frac{\hbar}{m}$ then:

$$\sigma_{x_\neg} \sigma_{p_\neg} \geq \frac{\hbar}{2m}.$$

In flat space, this reduces to the usual uncertainties between position and momentum, or time and energy.

Another uncertainty relation is $\sigma_\tau \sigma_{-\frac{\mu^2}{2}} \geq \frac{\hbar}{2m}$. This uncertainty relation is not truly an uncertainty relation since $\tau$ is an exact evolution parameter. This makes sense in analogy with the energy-time uncertainty relation from Schrödinger theory. The self energy and the self time have a conjugate relationship, though they are not directly conjugate.

# Chapter 8

# Flat Space

In this section, I shall find and describe solutions to the flat space metric $g_{\alpha\beta} = \eta_{\neg\alpha\beta}$ in Cartesian background coordinates. The relativistic equation requires the inverse of the flat metric:

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Plugging this inverse metric into the flat metric relativistic equation I obtain the partial differential equation:

$$\left(i\frac{\hbar}{m}\frac{\partial}{\partial\tau}\right)\Psi = -\frac{1}{2}\left(-i\frac{\hbar}{m}\frac{\partial}{\partial x^{\beta}}\right)\left(g^{\alpha\beta}\left(-i\frac{\hbar}{m}\frac{\partial}{\partial x^{\alpha}}\right)\right)\Psi.$$

Here $\hbar$ is Planck's constant and $m$ is the rest mass of the particle. A solution of this equation can be obtained by separation of variables:

$$\begin{aligned}
\widehat{-\frac{1}{2}p_\alpha g^{\alpha\beta}p_\beta} &= \frac{\hbar}{m}i\frac{\partial}{\partial\tau} = -\frac{\mu^2}{2}, \\
\widehat{p_t} &= -\frac{\hbar}{m}i\frac{\partial}{\partial x^t} = P_t, \\
\widehat{p_a} &= -\frac{\hbar}{m}i\frac{\partial}{\partial x^a} = P_a, \\
\widehat{p^t} &= \frac{\{p_t, g^{tt}\}}{2} = -\frac{\hbar}{m}i\frac{\partial}{\partial x^a} = P_t \sim \frac{\partial x^t}{\partial\tau}, \\
\widehat{p^a} &= \frac{\{p_a, g^{aa}\}}{2} = \frac{\hbar}{m}i\frac{\partial}{\partial x^a} = -P_a = P^a \sim \frac{\partial x^a}{\partial\tau}.
\end{aligned}$$

Here the relation ˜ means "interpreted as". The eigenvalue equation becomes, $\left(-\frac{\mu^2}{2} = -\frac{1}{2}\left(P_t^2 - P_a^2\right)\right)\Psi_{P_\alpha,\mu}$. In flat space, this can be rearranged into a statement of the fact that $E^2 = m^2 + p^2$. $E$ is the energy, $m$ is the rest mass and $p$ is the momentum.

An eigenfunction is thus,

$$\Psi_{P_\alpha,\mu}(x, y, z, t; \tau) = e^{i\frac{m}{\hbar}\left(P_\alpha x^\alpha + \frac{M^2}{2}\tau\right)} = e^{i\frac{m}{\hbar}\left(-P^a x^a + P_t t + \frac{\mu^2}{2}\tau\right)}$$

with $P_t^2 = \mu^2 + P_a^2$.

Example: Discuss the solution for a particle of known mass at rest.

For $\mu = 1$ and $P_t = 1$, then $P_a = 0$. The particle is at zero speed at a known mass, and the solution becomes:

$$\Psi_{\mu=1,P_t=1,P_a=0}(x, y, z, t; \tau) = e^{i\frac{m}{\hbar}\left(t+\frac{1}{2}\tau\right)}.$$

The particle has equal probability of being anywhere in space because its momentum is exactly 0 and anywhere in time because its energy is exactly the rest energy. Instead of starting to spread out, it is spread out over all space-time and I know nothing of its space-time point.

The expectation of the proper-Lagrangian is found to be:

$$\left\langle m^2\widehat{p_\tau}\right\rangle = \left\langle m^2\frac{\hbar}{m}i\frac{\partial}{\partial\tau}\right\rangle = -\frac{m^2}{2}.$$

The expectation of the energy of the particle is:

$$\langle Energy \rangle = \left\langle m\widehat{p^t} \right\rangle = \left\langle m \left( -\frac{\hbar}{m} i \frac{\partial}{\partial t} \right) \right\rangle = m.$$

Since the energy is known exactly, I could easily find $m\frac{dx^x}{dx^t}$ :

$$\left\langle m\frac{dx}{dt} \right\rangle = \left\langle m\widehat{p^x}\frac{1}{P^t} \right\rangle = 0.$$

Example: Discuss the solution for a quantum particle travelling upwards ($+z$) at velocity $\frac{3}{5}$ with energy $P^t = \frac{5}{4}$.

I can write the rest mass ratio immediately:

$$\begin{aligned}
\mu &= \sqrt{P_t^2 - P_z^2} = 1. \\
\Psi_{P^z=\frac{3}{4},P^t=\frac{5}{4},\mu=1}(x,y,z,t;\tau) &= e^{i\frac{m}{\hbar}\left(-\frac{3}{4}z+\frac{5}{4}t+\frac{1}{2}\tau\right)}. \\
\left\| \Psi_{P^z=\frac{3}{4},P^t=\frac{5}{4},\mu=1} \right\|^2 &= 1.
\end{aligned}$$

This particle yet again has no well defined space-time position. The particle is uniformly distributed over all space-time. The momentum is:

$$\left\langle m\frac{dx}{dt} \right\rangle = \left\langle m\widehat{p^x}\frac{1}{P^t} \right\rangle = \left\langle m \left( \frac{\hbar}{m} i \frac{\partial}{\partial x} \right) \frac{4}{5} \right\rangle = \frac{3}{5}m.$$

# Chapter 9

# Harmonic Oscillator

As another example, I solve the harmonic oscillator in one dimension. The potential energy of the harmonic oscillator is:

$$V = \frac{1}{2} m_t \omega^2 x^2.$$

Here $m_t$ is the test mass, $\omega$ is the classical frequency, $x$ is the space coordinate, and $V(x)$ is the potential energy function.

Putting this into the metric-potential formulation I obtain,

$$g_{\alpha\beta} = \begin{bmatrix} \left(\frac{1}{2}\omega^2 x^2 + 1\right)^2 & 0 \\ 0 & -\left(\frac{1}{2}\omega^2 x^2 + 1\right) \end{bmatrix},$$

where the metric $g_{\alpha\beta} = g_{\neg\alpha\beta}$ is the harmonic oscillator metric described in Cartesian background reference-frame coordinates.

The inverse of this metric is:

$$g^{\alpha\beta} = \begin{bmatrix} \left(\frac{1}{2}\omega^2 x^2 + 1\right)^{-2} & 0 \\ 0 & -\left(\frac{1}{2}\omega^2 x^2 + 1\right)^{-1} \end{bmatrix}.$$

Plugging this into the flat background metric relativistic quantum equation, I obtain:

$$-\frac{\mu^2}{2} X = -\frac{1}{2}\left(\left(\frac{1}{2}\omega^2 x^2 + 1\right)^{-2} P_t^2 - \widehat{p_x}\left(\frac{1}{2}\omega^2 x^2 + 1\right)^{-1} \widehat{p_x}\right) X.$$

Here I have made the substitutions $\widehat{p_\tau}\Psi_{\mu,P_t} = -\frac{\mu^2}{2}\Psi$, and $\widehat{p_t}\Psi_{\mu,P_t} = P_t\Psi$. Energy is going to be quantized in the harmonic oscillator. The solution is then of the form:

$$\Psi\left(x,t;\tau\right) = A\sum_{\mu,P_t} X\left(x\right) e^{i\frac{m}{\hbar}\left(P_t t+\frac{\mu^2}{2}\tau\right)}.$$

The energy $\frac{dt}{d\tau}$ with respect to the background coordinates is given by:

$$\widehat{p^t} = \frac{\left\{\widehat{p_t}, \left(\frac{1}{2}\omega^2x^2+1\right)^{-2}\right\}}{2} = \left(\frac{1}{2}\omega^2x^2+1\right)^{-2}\widehat{p_t}.$$

The velocity $\left(\frac{dx}{d\tau}\right)$ operator is given by:

$$\widehat{p^x} = \frac{\left\{\widehat{p_x}, g^{xx}\right\}}{2} = \frac{\left\{\widehat{p_x}, -\left(\frac{1}{2}\omega^2x^2+1\right)^{-1}\right\}}{2}.$$

# Chapter 10

# Hydrogen Atom and Black Hole

According to the metric-potential formulation, I can write the inverse metric times Jacobian product $g^{\alpha\beta}\sqrt{-\eta}$ of a spherical potential in spherical coordinates $(t, r, \theta, \phi)$ with Jacobian from Cartesian Lorentzian coordinates $\sqrt{-\eta}$, and general relativistic equation as:

$$\begin{aligned}
\sqrt{-\eta} &= r^2 \sin\theta, \\
g^{\alpha\beta}\sqrt{-\eta} &= \begin{bmatrix} \frac{r^2\sin\theta}{(U+1)^2} & 0 & 0 & 0 \\ 0 & -\frac{r^2\sin\theta}{(U+1)} & 0 & 0 \\ 0 & 0 & -\left(\sin\theta\left(U+1\right)\right)^{-1} & 0 \\ 0 & 0 & 0 & -\frac{\sin\theta}{(U+1)} \end{bmatrix}, \\
i\frac{\hbar}{m_q}\frac{\partial}{\partial\tau}\Psi &= -\frac{1}{2}\left(\sqrt{-\eta}\right)^{-1}\widehat{p_\alpha}\left(g^{\alpha\beta}\sqrt{-\eta}\right)\widehat{p_\beta}.
\end{aligned}$$

Separation of variables leads me to the usual angular solution:

$$Y_l^m(\theta,\phi) = \varepsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-\|m\|)!}{(l+\|m\|)!}}e^{im\phi}P_l^m(\cos\theta).$$

Here $P_l^m$ are the associated Legendre polynomials, $l$ is the azimuthal quantum number, $m$ is the magnetic quantum number, and the factor $\varepsilon = (-1)^m$ for $m > 0$ and $\varepsilon = 1$ for $m \leq 0$.

The radial equation for the potential $U = -\frac{\alpha}{r}$ (potential energy is $-\frac{m_t\alpha}{r}$ where $m_t$ is the test mass of the particle) reduces to:

$$\left(\frac{\partial}{\partial r^2} = \frac{l\,(l+1)}{r^2} - \frac{\alpha}{r^2\,(r-\alpha)} + \frac{\alpha}{r\,(r-\alpha)}\frac{\partial}{\partial r} - \left(\frac{m_q P_t}{\hbar}\right)^2 \frac{r}{r-\alpha} + \left(\frac{m_q \mu}{\hbar}\right)^2 \frac{r-\alpha}{r}\right) u\,(r)\,.$$

Here $R = \frac{u(r)}{r}$ as in the classical treatment of the hydrogen atom, $\alpha$ is the fine structure constant off by a constant factor, $m_q$ is the rest mass of the particle, and $r$ is the radial distance in flat spherical background coordinates.

Notice that the hydrogen radial equation is now $\frac{1}{r^3}$ dependent with a shift by the fine structure constant(or black hole radius in the case of gravity) in units of mass, $m_t$. The radial equation now depends on the radial velocity as seen in the third term. The surface of the black hole is at distance $r = \alpha$ and this causes us to consider three limits in the solution of a hydrogen atom: $r = 0, r = \alpha$, and $r \to \infty$. In the case of radiation from a black hole set $\mu = 0$ for massless particles.

An eigenfunction to the relativistic equation of a spin 0 hydrogen atom is:

$$\Psi_{P_t,l,m,\mu}\,(t, r, \theta, \phi; \tau) = \frac{1}{r} u Y_l^m e^{i\frac{m_q \mu^2}{2\hbar}\tau}.$$

The operators should be derived from the flat Cartesian operators in the obvious manner.

# Chapter 11

# Conclusion

I worked with the Einsteinian and Newtonian Lagrangians to get the relativistic metric in terms of the Newtonian four potentials.

This thesis developed novel and very reusable theories of general relativity and relativistic quantum mechanics. The theory of general relativity presented in this paper is derived directly from Lagrange's action which makes the physicist's transition from the study of Newtonian mechanics to general relativity more direct.

The metric is found to be an algebraic substitution function of the Newtonian vector potential. This metric is furthermore addible with other potential metrics which makes more complex metrics immediately accessible.

Magnetism was found to be an electromagnetic as well as a gravitational effect caused by a change of reference frame. I derived the metric and potential of a magnetic field of an electric current carrying wire by boosting from an electrically charged wire. This electromagnetic metric gives a different perspective of electromagnetism for relativistic particles.

The general relativity theory developed in this paper is one of the first theories to unify both the electromagnetic and gravitational fields into a symmetric metric.

I started from Einstein's Lagrangian to derive a Schrödinger wave function which evolves according to proper-time and which is an eigenfunction of the self-energy (e.g. rest mass) of the particle.

The symmetry of the metric is a very important property which enabled the derivation of the relativistic quantum equation. The relativistic quantum equation can be used to find the space-time wave function of a quantum particle. This wave-function evolves according to its proper-time in analogy with Schrödinger's space wave-function of a quantum particle that evolves according to an absolute time. The mathematical theory in this paper can be used to solve general relativistic quantum problems using equivalent tools from the Schrödinger equation. The established Schrödinger equation and its associated operator theory have their analogues in my general relativistic theory. My theory now adds a rest mass operator which gives predictions of a particle's mass state which will prove useful in elementary particle physics.

These two separate theories come together to create a versatile theory of quantum relativity. The metric from the general relativity theory is plugged into the quantum relativity equation.

Plugging the proton potential into the metric, I find the quantum radial equation of an electron orbiting a stationary proton that has no gravitational field, but an electric field. Spin is not considered in this thesis due to a lack of development time.

This thesis leaves room for further development, which I plan to do immediately. Inputting a simple travelling gravitational wave potential and an electromagnetic proton potential into my metric-potential formulation will produce an accurate model of a hydrogen atom affected by a gravitational wave. I will study the effect of a gravitational wave on the hydrogen atom using computer calculations. The calculated wavefunction can be used to study the effect of gravitational waves on atomic clock accuracy. I will also add spin to both theories.

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